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On the $SO(N)$ and $Sp(N)$ free energy of a closed oriented 3-manifold

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1. Introduction

Let G_N be a compact Lie group parameterized by N such as $SU(N)$, $SO(N)$ or $Sp(N)$, and let $\mathfrak{g}_N$ be the Lie algebra of G_N. The LMO invariant $Z_M \in \mathcal{A}(\emptyset)$ [4] of a closed 3-manifold M is presented by a linear sum of (a kind of) trivalent graphs, where $\mathcal{A}(\emptyset)$ denotes the $\mathbb{Q}$ vector space spanned by such trivalent graphs (subject to some relations). The $\mathfrak{g}_N$ weight system $W_{\mathfrak{g}_N}$ is a map $\mathcal{A}(\emptyset) \to \mathbb{Q}[[h]]$ such that $W_{\mathfrak{g}_N}(D)$ of a trivalent graph D of degree d is defined to be h^d times some polynomial in N of degree $\leq d+2$. When we fix a value of N, $W_{\mathfrak{g}_N}(\log Z_M)$ is a power series in h with $\mathbb{Q}$ coefficients. When we regard N as a variable, the weight system can be regarded as a map $W_{\mathfrak{g}_\star} : \mathcal{A}(\emptyset) \to \mathbb{Q}[N][[h]]$, and $W_{\mathfrak{g}_\star}(\log Z_M)$ is a power series in h whose coefficients are polynomials in N. Putting τ to be Nh if $G_N = SU(N)$, $(N-1)h$ if $G_N = SO(N)$, and $(N+1)h$ if $G_N = Sp(N)$, $W_{\mathfrak{g}_\star}(\log Z_M)$ is a power series in τ and h. We denote it by $F_M^{G_N}(\tau, h) \in h^{-2}\mathbb{Q}[[\tau, h]]$, and call it the G_N *free energy* of M [2]. Further, we put the coefficient of h^{g-2} in $F_M^{G_N}(\tau, h)$ to be $F_{M,g}^{G_N}(\tau) \in \mathbb{Q}[[\tau]]$, *i.e.*,

$$F_M^{G_N}(\tau, h) = \sum_{g=0}^{\infty} h^{g-2} F_{M,g}^{G_N}(\tau),$$

where the value of g implies the genus of some surface appearing in the definition of the weight system.

In this article, when $G_N = SO(N)$ and $Sp(N)$, we give an explicit presentation of the G_N free energy for lens spaces, and show that $F_{L(d,b),g}^{G_N}(\tau)$ of the lens space $L(d,b)$ is analytic in a neighborhood of zero, where we can choose the neighborhood independently of g. This analyticity has been conjectured by S. Garoufalidis, T.T.Q. Le and M. Mariño [2]. Moreover, we show that for any g, the genus g terms of $SO(N)$ and $Sp(N)$ free energy agree up to sign.

2. Definitions

We briefly review the LMO invariant Z_M of a closed oriented 3-manifold M, constructed by T.T.Q. Le, J. Murakami and T. Ohtsuki in [4]. We denote by $\mathcal{A}(\emptyset)$ the vector space over $\mathbb{Q}$ spanned by trivalent graphs whose vertices are oriented, modulo the AS, IHX and STU relations and denote by $\mathcal{A}(\emptyset)_{\text{conn}}$ the subspace of $\mathcal{A}(\emptyset)$ spanned by connected trivalent graphs. The degree of a trivalent graph is half the number of vertices. The LMO invariant Z_M takes values in $\mathcal{A}(\emptyset)$. It is known that $\log Z_M$ takes values in $\mathcal{A}(\emptyset)_{\text{conn}}$.

Let us recall the weight system associated with a semi-simple Lie algebra $\mathfrak{g}$. It is known that for a semi-simple Lie algebra $\mathfrak{g}$, one obtains a $\mathbb{Q}$ linear map $W_{\mathfrak{g}} : \mathcal{A}(\emptyset) \to \mathbb{Q}[[h]]$, called the weight system associated with $\mathfrak{g}$ (for general references, see [1, 5]). From a trivalent graph D of degree d in $\mathcal{A}(\emptyset)$, $W_{\mathfrak{g}}(D)$ is obtained by substituting $\mathfrak{g}$ into D, contracting a tensor at vertices and multiplying by h^d. When $\mathfrak{g} = \mathfrak{g}_N = \mathfrak{sl}_N, \mathfrak{so}_N$ or $\mathfrak{sp}_N$, regarding N as a variable, $W_{\mathfrak{g}_N}(D)$ of a connected trivalent graph D of degree d is h^d times some polynomial in N of degree $\leq d+2$ by Lemma 1 below, and we regard the weight system $W_{\mathfrak{g}_N}$ as a map $W_{\mathfrak{g}_*} : \mathcal{A}(\emptyset) \to \mathbb{Q}[N][[h]]$.

Lemma 1. *For $\mathfrak{g}_N = \mathfrak{sl}_N, \mathfrak{so}_N, \mathfrak{sp}_N$ and a connected trivalent graph D of degree d, $W_{\mathfrak{g}_N}(D)$ can be presented in the following form,*

$$W_{\mathfrak{g}_N}(D) = \sum_{0 \leq g \leq d+1} a_{\mathfrak{g}_N,g}(D) N^{d+2-g} h^d, \tag{1}$$

for some $a_{\mathfrak{g}_N,g}(D) \in \mathbb{Z}$.

Let G_N be a simple compact Lie group $SU(N)$, $SO(N)$ or $Sp(N)$ and let $\mathfrak{g}_N$ be the Lie algebra of G_N. Putting τ to be Nh for $\mathfrak{g} = \mathfrak{sl}$, $(N-1)h$ for $\mathfrak{g} = \mathfrak{so}$, and $(N+1)h$ for $\mathfrak{g} = \mathfrak{sp}$, $W_{\mathfrak{g}_*}(D)$ has the following form,

$$W_{\mathfrak{g}_*}(D) = \sum_{0 \leq g \leq d+1} c_{\mathfrak{g},g}(D) \tau^{d+2-g} h^{g-2}, \tag{2}$$

for some $c_{\mathfrak{g},g}(D) \in \mathbb{Z}$. Since $\log Z_M \in A(\emptyset)_{conn}$, $W_{\mathfrak{g}_*}(\log Z_M)$ can be presented in the following form,

$$W_{\mathfrak{g}_*}(\log Z_M) = \sum_{d>0} \sum_{0 \leq g \leq d+1} c_{\mathfrak{g},d,g}(M) \tau^{d+2-g} h^{g-2} \in h^{-2}\mathbb{Q}[[\tau, h]], \tag{3}$$

for some $c_{\mathfrak{g},d,g}(M) \in \mathbb{Q}$. As in [2], we define the G_N *free energy* of a rational homology 3-sphere M by

$$F_M^{G_N}(\tau, h) := W_{\mathfrak{g}_*}(\log Z_M) \in h^{-2}\mathbb{Q}[[\tau, h]],$$

and put the coefficient of h^{g-2} in $F_M^{G_N}(\tau,h)$ to be $F_{M,g}^{G_N}(\tau)\in\mathbb{Q}[[\tau]]$, *i.e.*,

$$F_M^{G_N}(\tau,h)=\sum_{g=0}^{\infty}F_{M,g}^{G_N}(\tau)h^{g-2}.$$

3. Results

We state the main theorem.

Theorem 1. *The $SO(N)$ and $Sp(N)$ free energy of the lens space $L(d,b)$ is presented by*

$$\begin{aligned}
&F_{L(d,b),g}^{G_N}(\tau)\\
&=\begin{cases}
\frac{1}{2}\{(g-1)\frac{B_g}{g!}(d^{2-g}\mathrm{Li}_{3-g}(e^{\tau/d})-\mathrm{Li}_{3-g}(e^{\tau}))+a_g(\tau)\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{if } g \text{ is even},\\
\varepsilon_{G_N}\left[\frac{(2^{g-2}-1)B_{g-1}}{(g-1)!}\left\{d^{2-g}(2^{2-g}\mathrm{Li}_{3-g}(e^{\tau/2d})-\frac{1}{2}\mathrm{Li}_{3-g}(e^{\tau/d}))\right.\right.\\
\qquad\left.\left.-2^{2-g}\mathrm{Li}_{3-g}(e^{\tau/2})+\frac{1}{2}\mathrm{Li}_{3-g}(e^{\tau})\right\}+a'_g(\tau)\right]\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{if } g \text{ is odd},
\end{cases}
\end{aligned}$$

where ε_{G_N} is 1 for $G_N=SO(N)$ and -1 for $G_N=Sp(N)$,

$$\begin{aligned}
&a_g(\tau)\\
&=\begin{cases}
-\frac{\tau^3}{12}(d^{-1}-1)-\frac{\pi^2\tau}{6}(d-1)+\frac{\tau^2}{2}\log d+(d^2-1)\zeta(3)+\lambda_{L(d,b)}\frac{\tau^3}{2}\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{if } g=0,\\
-\frac{\tau}{24}(d^{-1}-1)+\frac{1}{12}\log d-\lambda_{L(d,b)}\frac{\tau}{2} & \text{if } g=2,\\
0 & \text{if } g\geq 4,
\end{cases}
\end{aligned}$$

$$a'_g(\tau)=\begin{cases}
\frac{\tau}{2}\log d-\frac{\pi^2}{4}(d-1) & \text{if } g=1,\\
0 & \text{if } g\geq 3.
\end{cases}$$

Here the kth Bernoulli number B_k is defined by the generating series

$$\frac{x}{e^x-1}=\sum_{k=0}^{\infty}B_k\frac{x^k}{k!},$$

and the polylogarithm function Li_p *is defined by*

$$\mathrm{Li}_p(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^p}$$

for any integer p *and* $\zeta(3) := \sum_{n=1}^{\infty} \frac{1}{n^3}$. *In particular,* $F^{SO(N)}_{L(d,b),g}(\tau)$ *and* $F^{Sp(N)}_{L(d,b),g}(\tau)$ *are analytic in a neighborhood at zero, where we can choose the neighborhood independently of* g.

Outline of a proof of Theorem 1

Let Ψ_+ be the set of positive roots of $\mathfrak{g}$ and $|\Psi_+|$ the number of positive roots. We denote by $C_{\mathfrak{g}}$ the quadratic Casimir of $\mathfrak{g}$ and by $\dim \mathfrak{g}$ the dimension of $\mathfrak{g}$.

From [2], we have

$$(4)\quad F^{G_N}_{L(d,b)}(\tau, h) = \frac{\lambda_{L(d,b)}}{4} C_{\mathfrak{g}_N} \cdot \dim \mathfrak{g}_N \cdot h + \sum_{\alpha \in \Psi_+} \left(f((\alpha,\rho)h/d) - f((\alpha,\rho)h)\right),$$

where we define the function f by

$$f(x) := \log\left(\frac{\sinh(x/2)}{x/2}\right).$$

We consider the case $SO(N)$ with even N. The first term in the formula (4) is given by

$$\frac{\lambda_{L(d,b)}}{4} C_{\mathfrak{so}_N} \cdot \dim \mathfrak{so}_N \cdot h = \frac{\lambda_{L(d,b)}}{4} N(N-1)(N-2)h = \frac{\lambda_{L(d,b)}}{4}\left(\frac{\tau^3}{h^2} - \tau\right).$$

We calculate the second term of the right-hand side of (4). From the definition of sinh, we have the following presentation of $f(x)$,

$$(5)\qquad f(x) = \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k)!} x^{2k},$$

where B_k is the kth Bernoulli number. So, it follows that

$$\begin{aligned}
&\sum_{\alpha \in \Psi_+} f((\alpha,\rho)h) \\
&= \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k)!} h^{2k} \sum_{1 \le j \le 2n-2} \frac{2n-j-1}{2} j^{2k}
\end{aligned}$$

$$+\sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k)!}h^{2k}(1-2^{2k-1})\sum_{1\le j\le n-1}j^{2k}$$

$$=\frac{1}{2}\sum_{s=0}^{\infty}\frac{(1-2s)B_{2s}h^{2s-2}}{(2s)!}F_s^{even}(\tau)$$

$$+\sum_{s=0}^{\infty}\frac{(1-2^{2s-1})B_{2s}}{(2s)!}h^{2s-1}(2^{1-2s}F_s^{odd}(\tau/2)-\frac{1}{2}F_s^{odd}(\tau)),$$

where

$$F_s^{even}(\tau):=\sum_{l=0}^{\infty}\frac{B_{2l+2s}}{(2l+2s)(2l+2)!}\tau^{2l+2},$$

$$F_s^{odd}(\tau):=\sum_{l=0}^{\infty}\frac{B_{2l+2s}}{(2l+2s)(2l+1)!}\tau^{2l+1}.$$

It holds that $F_1^{even}(\tau)=f(\tau)$ and that $\frac{d}{d\tau}F_s^{even}(\tau)=F_s^{odd}(\tau)$. From the fact that the right hand side in the equation (5) is analytic function in the unit disk, we see that for any g, the power series $F_{M,g}^{G}(\tau)$ is analytic in the unit disk. Moreover, from the equation

$$f(\tau)=-\mathrm{Li}_1(e^\tau)-\frac{\tau}{2}-\log(-\tau)$$

in the unit disk [2] and the fact that

$$\frac{d}{dx}\mathrm{Li}_\alpha(e^x)=\mathrm{Li}_{\alpha-1}(e^x),$$

using a similar way in [2], we obtain

$$F_s^{even}(\tau)$$

$$=-\mathrm{Li}_{3-2s}(e^\tau)+\begin{cases}-\frac{\tau^2}{2}\log(-\tau)-\frac{\tau^3}{12}+\frac{3\tau^2}{4}-\frac{\pi^2\tau}{6}+\zeta(3) & \text{if } s=0,\\ -\log(-\tau)-\frac{\tau}{2} & \text{if } s=1,\\ (2s-3)!\tau^{2-2s}-\frac{B_{2s-2}}{2s-2} & \text{if } s\ge 2,\end{cases}$$

and

$$F_s^{odd}(\tau)=-\mathrm{Li}_{2-2s}(e^\tau)+\begin{cases}-\tau\log(-\tau)-\frac{1}{4}\tau^2-\frac{\pi^2}{6}+\tau & \text{if } s=0,\\ -\frac{1}{\tau}-\frac{1}{2} & \text{if } s=1,\\ -(2s-2)!\tau^{1-2s} & \text{if } s\ge 2.\end{cases}$$

Substituting these into the formula (5), we get the formula for $G = SO(N)$ with even N. Similarly, the formula for $G = SO(N)$ with odd N can be obtained. The formula for $G = Sp(N)$ follows from Proposition 1 below. $\square$

Furthermore, one has

Proposition 1. *For any closed oriented 3-manifold M and any g,*

$$F_{M,g}^{Sp(N)}(\tau) = (-1)^g F_{M,g}^{SO(N)}(\tau),$$

Proof. Noting that $\tau = N-1$ for $\mathfrak{g} = \mathfrak{so}$ and that $\tau = N+1$ for $\mathfrak{g} = \mathfrak{sp}$, it follows from (2) that

$$W_{\mathfrak{sp}_*}(D) = \sum_{0\le g\le d+1} c_{\mathfrak{sp},g}(D)(N+1)^{d+2-g}h^{g-2},$$

$$W_{\mathfrak{so}_*}(D) = \sum_{0\le g\le d+1} c_{\mathfrak{so},g}(D)(N-1)^{d+2-g}h^{g-2}$$

for a connected trivalent graph D of degree d. Hence,

$$\begin{aligned}(-1)^d W_{\mathfrak{so}_*}(D)|_{N\to -N} &= (-1)^d \sum_{0\le g\le d+1} c_{\mathfrak{so},g}(D)(-N-1)^{d+2-g}h^{g-2} \\ &= \sum_{0\le g\le d+1} (-1)^g c_{\mathfrak{so},g}(D)(N+1)^{d+2-g}h^{g-2}.\end{aligned}$$

Comparing $W_{\mathfrak{sp}_*}(D)$ and $(-1)^d W_{\mathfrak{so}_*}(D)|_{N\to -N}$ by Proposition 2 below, we have

$$c_{\mathfrak{sp},g}(D) = (-1)^g c_{\mathfrak{so},g}(D)$$

for any g. Since $\log Z_M$ is a linear sum of such D, it follows from (3) that

$$c_{\mathfrak{sp},d,g}(M) = (-1)^g c_{\mathfrak{so},d,g}(M)$$

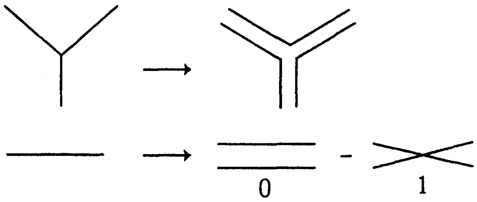
for any rational homology 3-sphere M, any d, and any g. Further, since

$$F_{M,g}^{G_N}(\tau) = \sum_{d>0, d\ge g-1} c_{\mathfrak{g},d,g}(M)\tau^{d+2-g}$$

by definition, we obtain the required formula. $\square$

Proposition 2. *For a connected trivalent graph D of degree d, $W_{\mathfrak{sp}_N}(D)$ is obtained from by replacing N to $-N$ in $W_{\mathfrak{so}_N}(D)$ and multiplying by $(-1)^d$, i.e., $W_{\mathfrak{sp}_N}(D) = (-1)^d W_{\mathfrak{so}_N}(D)|_{N\to -N}$.*

To give an outline of a proof of Proposition 2, we review results about $\mathfrak{so}_N$ and $\mathfrak{sp}_N$ weight systems. The following description of the weight system $W_{\mathfrak{so}_N}$ is known. We replace any trivalent vertex and any edge in the following:



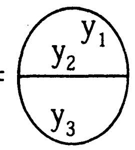
We denote by $e(D)$ the set of edges of a connected trivalent graph D. Given a map $m_e : e(D) \to \{0,1\}$, called a edge marking of D, choosing one of the two possibilities for the replacement of an edge depending on m_e, connecting up, we obtain an orientable or nonorientable surface S_{D,m_e} of the genus $g(S_{D,m_e})$ with b_{D,m_e} boundary components. Then, we have

$$W_{\mathfrak{so}_N}(D) = \sum_{m_e} (-1)^{s_{m_e}} N^{b_{D,m_e}} h^{g'_{D,m_e} - 2 + b_{D,m_e}}, \tag{6}$$

where $s_{m_e} = \sum_{y \in e(D)} m_e(y)$, the sum is over all possible edge markings m_e of D, and

$$g'_{D,m_e} = \begin{cases} 2g(S_{D,m_e}) & \text{if } S_{D,m_e} \text{is orientable} \\ g(S_{D,m_e}) & \text{if } S_{D,m_e} \text{is nonorientable.} \end{cases}$$

Example. We consider the following trivalent graph of degree 1

$D =$ 

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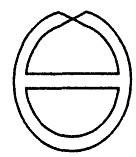
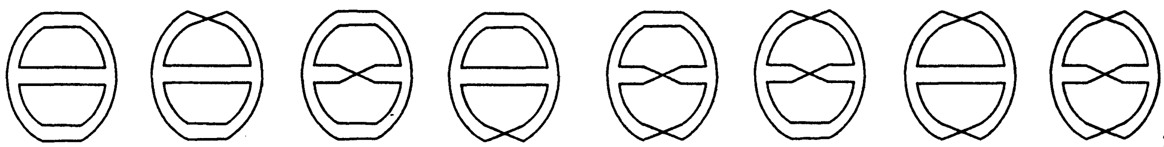
and the edge marking with $m_e(y_1) = 1$, $m_e(y_2) = 0$, and $m_e(y_3) = 0$. Then we get

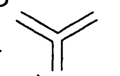
$S_{D,m_e} =$  ,

which is a projective plane with two boundary components. This contributes $-N^2 h$ to $W_{\mathfrak{so}_N}(D)$. From 8 possible edge markings, we get the following surfaces

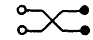

,

and obtain that $W_{\mathfrak{so}_N}(D) = N^3h - 3N^2h + 3Nh - Nh = N(N-1)(N-2)h$.

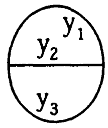
Next, we give a description of the weight system $W_{\mathfrak{sp}_N}$ with $N = 2n$. We denote by $e(D)$ the set of edges of a connected trivalent graph D and Y' the set of the diagrams

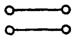
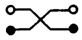
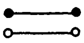
We replace any trivalent vertex in the same way as the weight system $W_{\mathfrak{so}_N}$ and replace each edge with one diagram in Y', in such a way that connecting up, the two ends of each arc in  have the same symbols. Such a replacement defines a map $m' : e(D) \to Y'$, called an admissible edge marking of D, and we obtain an orientable or nonorientable surface $S_{D,m'}$ of the genus $g(S_{D,m'})$ with $b_{D,m'}$ boundary components with even symbols $\circ$ and even symbols $\bullet$. Then, we have

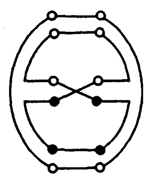
$$W_{\mathfrak{sp}_N}(D) = \sum_{m'} (-1)^{s_{m'}} n^{b_{D,m'}} h^{g'_{D,m'} - 2 + b_{D,m'}},$$

where $s_{m'}$ is the number of  and  in $S_{D,m'}$, the sum is over all possible admissible edge markings m' of D, and $g'_{D,m'} = 2g(S_{D,m'})$ if the surface $S_{D,m'}$ is orientable and $g'_{D,m'} = g(S_{D,m'})$ if the surface $S_{D,m'}$ is nonorientable.

Example. We consider the trivalent graph

$D =$ ,

and the admissible edge marking m' with $m'(y_1) =$ , $m'(y_2) =$ , and $m'(y_3) =$ . This gives a nonorientable surface

$S_{D,m'} =$ 

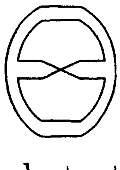
of the genus 1 with 2 boundary component and so contributes nh to $W_{\mathfrak{sp}_N}(D)$. We compute that $W_{\mathfrak{sp}_N}(D) = 8n^3h + 12n^2h + 4nh = 2n(2n+1)(2n+2)h = N(N+1)(N+2)h$.

Now let us give an idea of a proof of Proposition 2. From the above desctiption of $W_{\mathfrak{so}_N}$ and $W_{\mathfrak{sp}_N}$ with $N = 2n$, we have

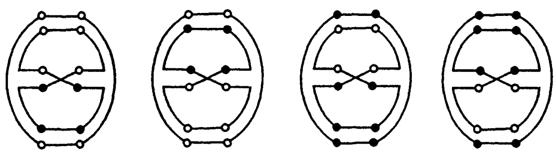
$$W_{\mathfrak{so}_N}(D) = \sum_{m_e} (-1)^{s_{m_e}} N^{b_{D,m_e}} h^d,$$

$$W_{\mathfrak{sp}_N}(D) = \sum_{m'} (-1)^{s_{m'}} n^{b_{D,m'}} h^d.$$

For example, we consider $D = \ominus$. We have that the surface



appearing in $W_{\mathfrak{so}_N}(\ominus)$ corresponds to the 4 surfaces



appearing in $W_{\mathfrak{sp}_N}(\ominus)$. We have that one surface appearing in $W_{\mathfrak{so}_N}(D)$ corresponds to $2^{b_{D,m_e}}$ surfaces appearing in $W_{\mathfrak{sp}_N}(D)$. Then, it follows that

$$W_{\mathfrak{sp}_N}(D) = \sum_{m_e} (-1)^{s_{m'}} 2^{b_{D,m_e}} n^{b_{D,m_e}} h^d.$$

Noting that $N = 2n$ and $s_{m'} \equiv s_{m_e} + d + b_{D,m_e} \pmod 2$, the claim holds.

References

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